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The Loschmidt Echo $M(t)$ (defined as the squared overlap of wave packets evolving with two slightly different Hamiltonians) is a measure of quantum reversibility. We investigate its behavior for classically quasi-integrable systems. A dominant regime emerges where $M(t) \propto t^{-\alpha}$ with $\alpha = 3d/2$ depending solely on the dimension d of the system. This power law decay is faster than the result $\propto t^{-d}$ for the decay of classical phase space densities.

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The search for quantum signatures of chaos has provided much insight into how classical dynamics manifests itself in quantum mechanics [1,2]. The basic question is how to determine from a system's quantum properties whether the classical limit of its dynamics is chaotic or regular. One very successful approach has been to look at the spectral statistics, in particular the distribution of level spacings [3]. An altogether different approach, advocated by Schack and Caves [4], has been to investigate the sensitivity of the quantum dynamics to perturbations of the Hamiltonian. This approach goes back to the early work of Peres [5] and has attracted new interest recently in connection with the study of decoherence and quantum reversibility [6–12].

The basic quantity in this approach is the so-called Loschmidt Echo, i.e. the fidelity

$$M(t) = |\langle\psi_0| \exp(iHt) \exp(-iH_0 t)|\psi_0\rangle|^2 \qquad (1)$$

with which a narrow wavepacket ψ_0 can be reconstructed by inverting the dynamics after a time t with a perturbed Hamiltonian $H = H_0 + V$ [5,6]. (We set $\hbar = 1$.) The fidelity quantifies the sensitivity of the time-reversal operation to the uncertainty in the Hamiltonian, and thus provides for a measure of quantum reversibility.

To date, most investigations of $M(t)$ focused on classically chaotic Hamiltonians H and H_0 [6–10]. One notable exception it the original paper by Peres [5], who noted that the decay of $M(t)$ is slower in a regular system — but did not quantify it further. We will show in this article that in a regular system a dominant regime emerges where $M(t)$ has a power law decay $\propto t^{-3d/2}$, with an exponent depending solely on the dimension d of the system. This power law decay establishes the higher degree of quantum reversibility of regular systems compared to chaotic ones, where $M(t)$ decays exponentially. This trend is as expected from classical reversibility (defined in terms of the decay of the overlap of classical phase space distributions [13]). However, we find that quantum mechanics plays a crucial role in regular systems by inducing a parametrically faster power law decay $\propto t^{-3d/2}$ than the classical one $\propto t^{-d}$.

We consider the generic situation of a regular or quasi-integrable H_0 and a perturbation potential V that has no common integral of motion with H_0. (By regular or quasi-integrable we mean systems with a phase space dominated by invariant tori, as in Fig. 1.) This condition ensures that, classically, the perturbation has a component transverse to the invariant tori almost everywhere in phase space, in contrast to recently studied nongeneric cases where either the perturbation V almost commutes with H_0, or H_0 and H belong to a continuous family of integrable Hamiltonians [11,12]. This explains why Refs. [11,12] arrived at the opposite conclusion that $M(t)$ decays faster in an integrable system than in a chaotic one.

We follow the semiclassical approach of Jalabert and Pastawski [6]. We start from a Gaussian wavepacket $\psi_0(\mathbf{r}'_0) = (\pi\sigma^2)^{-d/4} \exp[i\mathbf{p}_0 \cdot (\mathbf{r}'_0 - \mathbf{r}_0) - |\mathbf{r}'_0 - \mathbf{r}_0|^2/2\sigma^2]$ and approximate its time evolution by

$$\exp(-iHt)\psi_0(\mathbf{r}) = \int d\mathbf{r}'_0 \sum_s K_s^H(\mathbf{r}, \mathbf{r}'_0; t)\psi_0(\mathbf{r}'_0), \qquad (2)$$

$$K_s^H(\mathbf{r}, \mathbf{r}'_0; t) = C_s^{1/2} \exp[iS_s^H(\mathbf{r}, \mathbf{r}'_0; t) - i\pi\mu_s/2]. \qquad (3)$$

The semiclassical propagator is expressed as a sum over classical trajectories (labelled s) connecting $\mathbf{r}$ and $\mathbf{r}'_0$ in the time t. For each s, the partial propagator contains the action integral $S_s^H(\mathbf{r}, \mathbf{r}'_0; t)$ along s, a Maslov index μ_s (which will drop out), and the determinant C_s of the monodromy matrix. Since we consider a narrow initial wavepacket, we linearize the action in $\mathbf{r}'_0 - \mathbf{r}_0$ and perform the integration over $\mathbf{r}'_0$. After a stationary phase approximation, the semiclassical fidelity reads

$$M(t) = (4\pi\sigma^2)^d \left| \int d\mathbf{r} \sum_s K_s^H(\mathbf{r}, \mathbf{r}_0; t)^* K_s^{H_0}(\mathbf{r}, \mathbf{r}_0; t) \exp(-\sigma^2|\mathbf{p}_s - \mathbf{p}_0|^2) \right|^2, \qquad (4)$$

with initial momentum $\mathbf{p}_s = -\partial S_s/\partial \mathbf{r}_0$.

Eqs. (2–4) are equally valid for regular and chaotic Hamiltonians, as long as semiclassics applies. Squaring the amplitude in Eq. (4) leads to a double sum over classical paths s and s' and a double integration over coordinates $\mathbf{r}$

and $\mathbf{r}'$. Accordingly, $M(t) = M^{(\mathrm{d})}(t) + M^{(\mathrm{nd})}(t)$ splits into diagonal ($s = s'$) and nondiagonal ($s \neq s'$) contributions. The diagonal contribution sensitively depends on whether H_0 is regular or chaotic. Ref. [6] found that $M^{(\mathrm{d})}(t) \propto \exp(-\lambda t)$ for chaotic dynamics, with λ the Lyapunov exponent. We will show that the decay turns into a power law $M^{(\mathrm{d})}(t) \propto t^{-3d/2}$ for regular dynamics. The nondiagonal contribution, on the contrary, is insensitive to the nature of the classical dynamics (set by H_0), provided the perturbation Hamiltonian V has no common integral of motion with H_0. Ref. [7] found that $M^{(\mathrm{nd})}(t) \propto \exp(-\Gamma t)$ for chaotic dynamics, with Γ given by the golden rule spreading width of an eigenstate of H_0 over the eigenbasis of H. (This golden rule decay requires that Γ is larger than the level spacing Δ.) We will see that the same exponential decay of $M^{(\mathrm{nd})}(t)$ holds when H_0 is regular, so that $M^{(\mathrm{d})}(t)$ always dominates in the long time limit. Consequently, the fidelity decays exponentially, $\propto \exp[-\min(\Gamma, \lambda)t]$ for chaotic systems, while for regular systems the decay is algebraic, $\propto t^{-3d/2}$, as it is then set by the diagonal contribution. The golden rule width Γ still determines the regime of validity of the power law decay via the condition $\Gamma > \Delta$.

Continuing from Eq. (4), and still following Ref. [6], we write $M(t)$ as

$$M(t) = (4\pi\sigma^2)^d \int d\mathbf{r} \int d\mathbf{r}' \sum_{s,s'} C_s C_{s'} \exp[i\delta S_s(\mathbf{r}, \mathbf{r}_0; t) - i\delta S_{s'}(\mathbf{r}', \mathbf{r}_0; t)] \exp(-\sigma^2 |\mathbf{p}_s - \mathbf{p}_0|^2 - \sigma^2 |\mathbf{p}_{s'} - \mathbf{p}_0|^2), \tag{5}$$

with $\delta S_s(\mathbf{r}, \mathbf{r}_0; t) = S_s^H(\mathbf{r}, \mathbf{r}_0; t) - S_s^{H_0}(\mathbf{r}, \mathbf{r}_0; t)$. Considering first the diagonal contribution $M^{(\mathrm{d})}(t)$, we set $s = s'$ and expand the phase difference as

$$\delta S_s(\mathbf{r}, \mathbf{r}_0; t) - \delta S_s(\mathbf{r}', \mathbf{r}_0; t) = \int_0^t d\tilde{t}\; \nabla V[\mathbf{q}(\tilde{t})] \cdot \big(\mathbf{q}(\tilde{t}\,) - \mathbf{q}'(\tilde{t}\,)\big). \tag{6}$$

The points $\mathbf{q}$ and $\mathbf{q}'$ lie on the classical path with $\mathbf{q}(t) = \mathbf{r}$, $\mathbf{q}'(t) = \mathbf{r}'$, and $\mathbf{q}(0) = \mathbf{q}'(0) = \mathbf{r}_0$. In a regular system, the distance between two initially close points increases linearly with time, $|\mathbf{q}(\tilde{t}) - \mathbf{q}'(\tilde{t})| \simeq (\tilde{t}/t)|\mathbf{r} - \mathbf{r}'|$. Here we depart from the exponential divergence $\propto \exp[\lambda(\tilde{t} - t)]$ assumed in Ref. [6] for chaotic dynamics.

The spatial integrations and the sums over classical paths in Eq. (5) lead to the phase averaging

$$\exp(i\delta S_s - i\delta S_s') \rightarrow \langle \exp(i\delta S_s - i\delta S_s')\rangle \simeq \exp[-\tfrac{1}{2}\langle(\delta S_s - \delta S_s')^2\rangle]. \tag{7}$$

Since V and H_0 have no common integral of motion, we may expect a fast decay of the correlations, $\langle \partial_i V[\mathbf{q}(\tilde{t})] \partial_j V[\mathbf{q}(\tilde{t}')]\rangle = U\delta_{ij}\delta(\tilde{t} - \tilde{t}')$. One then gets

$$\begin{aligned} M^{(\mathrm{d})}(t) &= (4\pi\sigma^2)^d \int d\mathbf{r} \int d\mathbf{r}' \sum_s C_s^2 \exp\big(-\tfrac{1}{2}U \int_0^t d\tilde{t}\; (\tilde{t}/t)^2 |\mathbf{r} - \mathbf{r}'|^2\big) \exp(-2\sigma^2 |\mathbf{p}_s - \mathbf{p}_0|^2) \\ &= (4\pi\sigma^2)^d \int d\mathbf{r}_+ \int d\mathbf{r}_- \sum_s C_s^2 \exp\big(-\tfrac{1}{6}Ut\, \mathbf{r}_-^2\big) \exp(-2\sigma^2 |\mathbf{p}_s - \mathbf{p}_0|^2). \end{aligned} \tag{8}$$

The Gaussian integration over $\mathbf{r}_- \equiv \mathbf{r} - \mathbf{r}'$ ensures that $\mathbf{r} \approx \mathbf{r}'$, and hence $\mathbf{r}_+ \equiv (\mathbf{r} + \mathbf{r}')/2 \approx \mathbf{r}$. One C_s is then absorbed by a change of variable from $\mathbf{r}_+$ to $\mathbf{p}_s$, and the Gaussian integral over $\mathbf{r}_-$ gives a factor $\propto t^{-d/2}$. Finally, setting $C_s \approx t^{-d}$ as is the case in a regular system, we arrive at

$$M^{(\mathrm{d})}(t) \propto t^{-3d/2}, \tag{9}$$

which is the central result of this paper. The power law (9) holds once the perturbation is strong enough to induce a golden rule spreading of the eigenstates of H_0 over the eigenbasis of H (which is the range of validity [6,7] of the above semiclassical approach), and under the assumption that the perturbation potential varies rapidly along a classical trajectory of H_0. [We used this assumption to average the complex exponential in Eq. (7).]

The nondiagonal contribution ($s \neq s'$) to Eq. (5) is the same as in Refs. [6,7]. The phase averaging can be performed separately for s and s' and one gets

$$\langle \exp[i\delta S_s]\rangle = \exp(-\tfrac{1}{2}\langle \delta S_s^2\rangle) = \exp\left(-\tfrac{1}{2}\int_0^t d\tilde{t} \int_0^t d\tilde{t}' \langle V[\mathbf{q}(\tilde{t})] V[\mathbf{q}(\tilde{t}')]\rangle\right). \tag{10}$$

The point $\mathbf{q}(\tilde{t})$ lies on path s with $\mathbf{q}(0) = \mathbf{r}_0$ and $\mathbf{q}(t) = \mathbf{r}$. If V and H_0 have no common integral of motion, the correlator of V gives the golden rule decay $\propto \exp(-\Gamma t)$ regardless of whether H_0 is chaotic or regular [15]. We conclude that for regular systems, the fidelity is dominated by the algebraically decaying diagonal contribution.

In order to check numerically the analytical result (9), we have studied the kicked top Hamiltonian [1]

$$H_0 = (\pi/2\tau)S_y + (K/2S)S_z^2 \sum_n \delta(t - n\tau), \tag{11}$$

which describes a vector spin of conserved magnitude S, undergoing a free precession around the y-axis, which is periodically perturbed (period τ) by sudden rotations around the z-axis over an angle proportional to S_z. Because S is conserved, H_0 is a one-dimensional Hamiltonian ($d = 1$), with a two-dimensional classical phase space consisting of the sphere of radius $S = 1$. The canonically conjugated variables are $(\varphi, \cos\theta)$, where θ and φ are spherical coordinates.

The classical limit of the kicked top is given by the map [1]

$$\begin{cases} x_{n+1} = z_n \cos(K x_n) + y_n \sin(K x_n) \\ y_{n+1} = -z_n \sin(K x_n) + y_n \cos(K x_n) \\ z_{n+1} = -x_n, \end{cases} \tag{12}$$

in the cartesian coordinates $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, and $z = \cos\theta$. Depending on the kicking strength K, the classical dynamics is regular, partially chaotic, or fully chaotic. We consider a kicking strength $K = 1.1$ for which the dynamics is regular for most of phase space, as is illustrated by the Poincaré map in Fig. 1. We checked that our results are not sensitive to the value of K, as long as the dynamics remains regular.

The quantum mechanical time evolution after n periods is given by the n-th power of the Floquet operator

$$F_0 = \exp[-i(K/2S)S_z^2]\exp[-i(\pi/2)S_y]. \tag{13}$$

We perturb the reversed time evolution by a periodic rotation of constant angle around the x-axis, slightly delayed with respect to the kicks in H_0,

$$H_1 = \phi S_x \sum_n \delta(t - n\tau - \epsilon). \tag{14}$$

The corresponding Floquet operator is $F = \exp(-i\phi S_x)F_0$. We set $\tau = 1$ for ease of notation, and varied S between 250 and 1000 (both H and H_0 conserve the spin magnitude). We calculated the average decay $\overline{M}$ of $M(t = n) = |\langle\psi_0|(F^\dagger)^n F_0^n|\psi_0\rangle|^2$ taken over 50 to 200 initial Gaussian wavepackets (coherent states) ψ_0.

In Fig. 2 we show the decay of $\overline{M}$ for $S = 1000$ and different perturbation strengths ϕ. For weak perturbations, the decay of $\overline{M}$ is exponential, and not Gaussian as one would expect from first order perturbation theory [5]. The reason why the perturbation operator S_x gives no first order correction is that for $K = 1.1$, eigenfunctions of F_0 are almost identical to eigenfunctions of S_y, so that diagonal matrix elements of H_1 vanish in this basis. For weak ϕ, the local spectral density of states $\rho(\epsilon)$ consists then of a delta function at zero energy plus an algebraically decaying tail [14]. Because of the absence of a first-order correction, the decay of the fidelity is given by the Fourier transform of this tail [10]. We numerically obtained a decay $\rho(\epsilon) \propto (\epsilon^2 + \gamma^2/4)^{-1}$ with $\gamma \propto \phi^{1.5}$. The resulting exponential decay $\propto \exp(-\gamma t)$ of the fidelity differs from the golden rule decay $\propto \exp(-\Gamma t)$ with $\Gamma \propto \phi^2$. As ϕ increases, the decay of $\overline{M}$ turns into the predicted power law $\propto t^{-3/2}$ – and not into the Gaussian decay found in Ref. [11] for a nongeneric perturbation $\propto S_z^2$. The power law decay of $M(t)$ prevails as soon as one enters the golden rule regime, i.e. for $\Gamma/\Delta \approx \phi^2 S^3 \gtrsim 1$ [7]. One therefore expects the power law decay to appear as S is increased at fixed ϕ, which is indeed observed in the inset to Fig. 2.

It is instructive to contrast these results for the decay of the overlap of quantum wavefunctions with the decay of the overlap of classical phase space distributions, a "classical fidelity" problem that has recently been investigated [9,11,13]. We assume that the two phase space distributions ρ_0 and ρ are initially identical and evolve according to the Liouville equation of motion corresponding to the classical map (12) for two different Hamiltonians H_0 and H. We consider regular dynamics and ask for the decay of the normalized phase space overlap

$$M_c(t) = \int d\mathbf{x} \int d\mathbf{p}\ \rho_0(\mathbf{x}, \mathbf{p}; t)\ \rho(\mathbf{x}, \mathbf{p}; t)/N_\rho, \tag{15}$$

where $N_\rho = (\int d\mathbf{x} \int d\mathbf{p}\rho_0)^{1/2}(\int d\mathbf{x} \int d\mathbf{p}\rho)^{1/2}$.

We have found above that a factor $\propto t^{-d/2}$ in the decay of the quantum fidelity $M(t) \propto t^{-3d/2}$ originates from the action phase difference and is thus of purely quantum origin. One therefore expects a slower classical decay $M_c(t) \propto \underline{C_s} \propto t^{-d}$. In Fig. 3 we show the decay of the averaged $\overline{M}_c$ taken over 10^4 initial points within a narrow volume of phase space, for $K = 1.1$ and $\phi = 1.7 \cdot 10^{-4}$. The decay is $\overline{M}_c \propto t^{-1}$ as expected for $d = 1$, and clearly differs from the quantum decay $\propto t^{-3/2}$.

Our investigations of the Loschmidt Echo (1) in the generic regime of classically quasi-integrable dynamics show that its decay is dominated by the power law $M(t) \propto t^{-\alpha}$. While from purely classical considerations one expects an exponent $\alpha_c = d$, we semiclassically obtain an anomalous exponent $\alpha = 3d/2$. This is corroborated by numerical simulations. The power law decay is to be contrasted with the exponential decay found for chaotic systems, thereby providing for a novel "quantum signature of chaos".

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[1] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 2000).
[2] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer, New York, 1990).
[3] O. Bohigas, M.J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[4] R. Schack and C.M. Caves, Phys. Rev. Lett. **71**, 525 (1993).
[5] A. Peres, Phys. Rev. A **30**, 1610 (1984).
[6] R.A. Jalabert and H.M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001).
[7] Ph. Jacquod, P.G. Silvestrov, and C.W.J. Beenakker, Phys. Rev. E **64**, 055203 (R) (2001).
[8] N.R. Cerruti and S. Tomsovic, Phys. Rev. Lett. **88**, 054103 (2002); F.M. Cuchietti, C.H. Lewenkopf, E.R. Mucciolo, H.M. Pastawski, and R.O. Vallejos, nlin.CD/0112015; Z.P. Karkuszewski, C. Jarzynski, and W.H. Zurek, quant-ph/0111002; Ph. Jacquod, I. Adagideli, and C.W.J. Beenakker, nlin.CD/0203052.
[9] G. Benenti and G. Casati, quant-ph/0112060.
[10] D.A. Wisniacki and D. Cohen, quant-phys/0111125.
[11] T. Prosen and M. Znidaric, J. Phys. A **35**, 1455 (2002).
[12] T. Prosen and T. Seligman, nlin.CD/0201038.
[13] B. Eckhardt, to be published.
[14] D. Cohen and E.J. Heller, Phys. Rev. Lett. **84**, 2841 (2000).
[15] This conclusion, that the golden rule decay holds whether H_0 is regular or chaotic, can also be obtained via a fully quantum mechanical approach based on random-matrix theory assumptions for V. The invariance under unitary transformations of the distribution of V is sufficient to obtain the exponential decay $M^{(\mathrm{nd})}(t) \propto \exp(-\Gamma t)$, irrespective of the distribution of H_0.

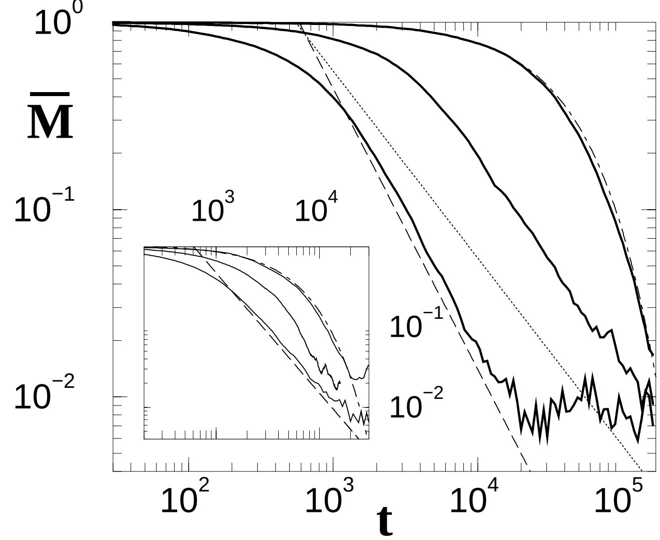
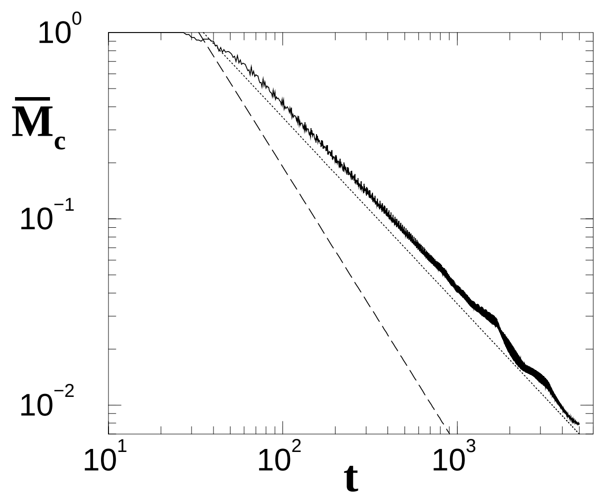


FIG. 2. Decay of $\overline{M}$ for $S = 1000$, $K = 1.1$, and $10^5\,\phi = 1.5$, 4.5, and 1
from exponential to power-law decay is illustrated by the dotted-dashed li
The dotted line gives the classical decay $\propto t^{-1}$. Inset: Decay of $\overline{M}$ for K
lines from right to left). The dashed and dotted-dashed lines indicate the
decay, respectively. These plots show that the $t^{-3/2}$ decay is reached eith
spin magnitude S, or by increasing S at fixed ϕ.



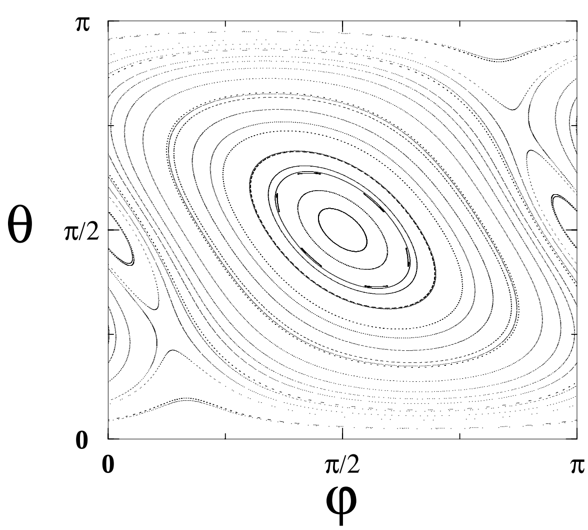


FIG. 3. Decay of the average overlap (15) of classical phase space d
$\phi = 1.7 \cdot 10^{-4}$ (solid line). The dotted and dashed lines give the classical
respectively.

FIG. 1. Poincaré map (12) for the classical kicked top at kicking strength $K = 1.1$.